6 Monte-Carlo Integration

Wednesday, March 05, 2014 2:29 PM

Monte-Carlo Integration

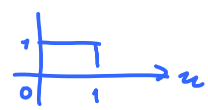
Recall: If X is a continuous RV with pdf $f_X(x)$,

then $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

this could be approximated by LLN : $\frac{1}{n}\sum_{k=1}^{n} g(X_k) \longrightarrow \mathbb{E}[g(X)]$

For example, consider $U \sim \mathcal{U}(0,1)$

$$f_U(u) = \begin{cases} 1, & 0<u<1, \\ 0, & \text{otherwise.} \end{cases}$$



$$\mathbb{E}[g(U)] = \int_{-\infty}^{\infty} g(u) f_U(u)\, du = \int_0^1 g(u)\, du$$

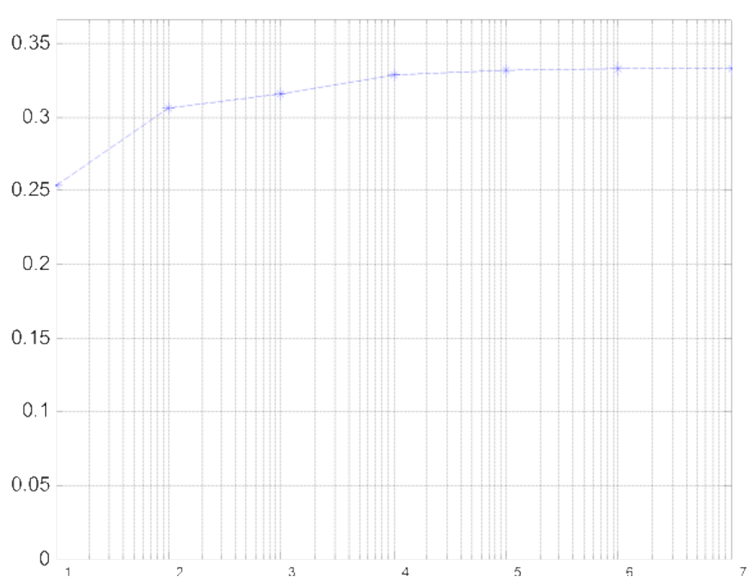
Ex. To find $\int_0^1 x^2\, dx$,

In this case $g(x) = x^2$

i) Generate many $U \sim \mathcal{U}(0,1)$

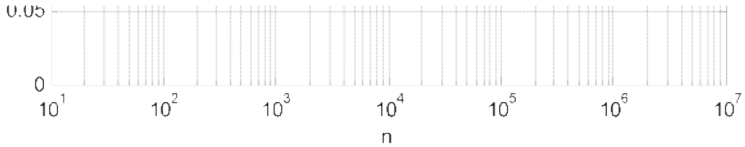
ii) Find the corresponding U^2

iii) Average



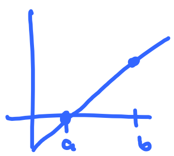
MonteCarloInt_Ex_1.m

```
%% In-class Example

close all; clear all;

g = @(x) x.^2; % Define the function to be integrated
a = 0; b = 1; % Limits for the integration

n = 7;
UU = rand(1,10^n);

%I = sum((b-a)*g((b-a)*UU+a))/(10^n)
I = mean((b-a)*g((b-a)*UU+a))

%% Visualizing the convergence
II = zeros(1,n);
for k = 1:n
   U = UU(1:10^k);
   II(k) = sum((b-a)*g((b-a)*U+a))/(10^k);
end

semilogx(10.^(1:n),II,'*--')
xlabel('n')
grid on
ylim([0 (1.1*max(II))])
```




```
xlabel('n')
grid on
ylim([0 (1.1*max(II))])

%% Check answer using symbolic math
syms x
I = int(g(x),x,0,1)
double(I)
```

Ex $\displaystyle\int_a^b g(x)\,dx = \int_0^1 g((b-a)y+a)\,(b-a)\,dy$



$y = \dfrac{x-a}{b-a} \rightarrow dy = \dfrac{dx}{b-a}$

$x = (b-a)y + a$

i) Generate many $U \sim \mathcal{U}(0,1)$

ii) Find the corresponding

$(b-a)\,g\big((b-a)U + a\big)$

iii) Average

Ex. $\displaystyle\int_0^\infty g(x)\,dx = \int_0^1 g\left(\frac{1}{y}-1\right)\frac{1}{y^2}\,dy$

$y = \dfrac{1}{x+1}$ $\qquad dy = -\dfrac{1}{(x+1)^2}\,dx = -y^2 dx$

$xy + y = 1$

$xy = 1 - y$

$x = \dfrac{1}{y} - 1$

$dx = -\dfrac{1}{y^2}\,dy$

i) Generate many $U \sim \mathcal{U}(0,1)$

ii) Find the corresponding

$$\frac{g\left(\frac{1}{U}-1\right)}{U^2}$$

iii) Average.

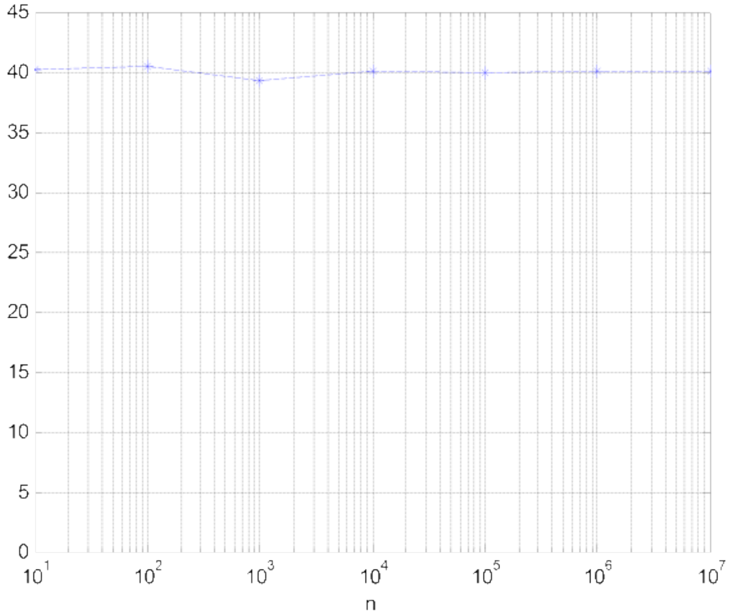
In-class Exercise :

Use Monte-Carlo Integration to evaluate the following integrals

a) $\displaystyle\int_1^3 x^3 + 3x + 4\,dx$ $\qquad$ b) $\displaystyle\int_1^3 e^{-x^2}\,dx$

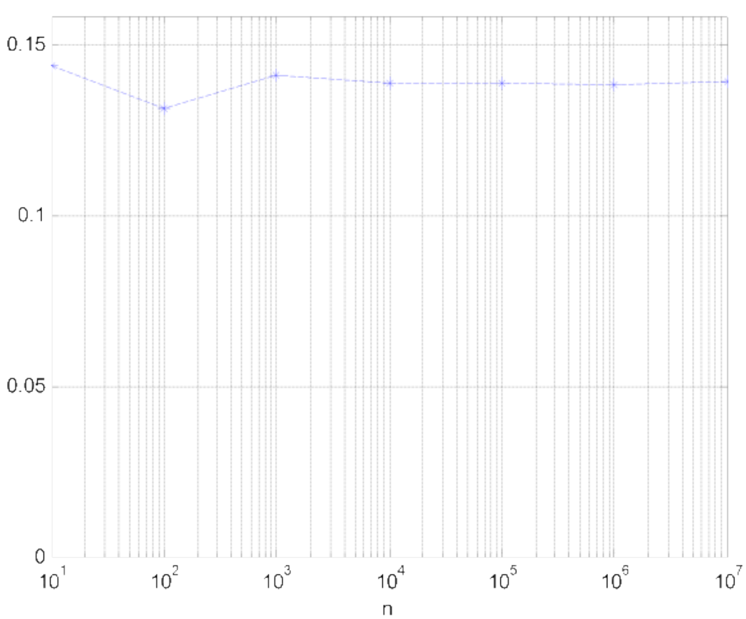
c) $\displaystyle\int_0^\infty e^{-x^2}\,dx$

Check your answers using MATLAB symbolic toolbox.



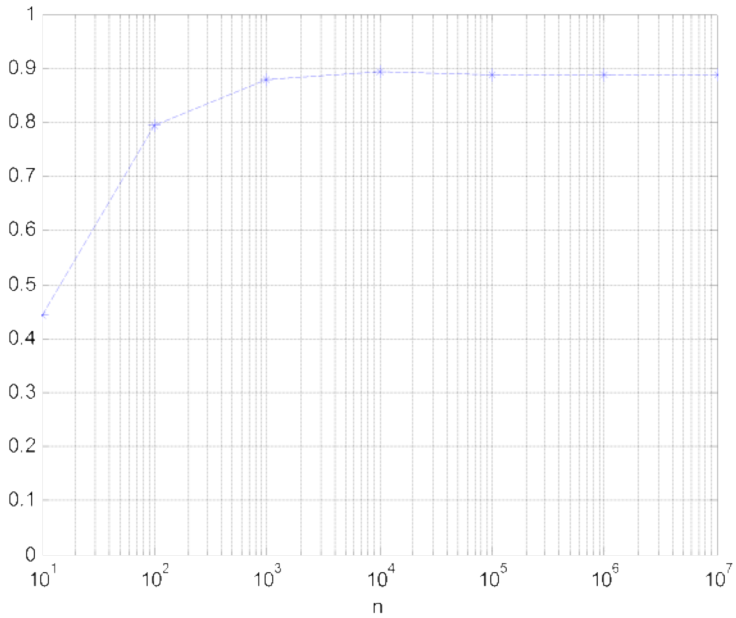

```
%% In-class Exercise (a)

close all; clear all;

g = @(x) x.^3 + 3*x + 4; % Define the function to be integrated
a = 1; b = 3; % Limits for the integration

n = 7;
UU = rand(1,10^n);

I = sum((b-a)*g((b-a)*UU+a))/(10^n)


%% Visualizing the convergence
II = zeros(1,n);
for k = 1:n
   U = UU(1:10^k);
   II(k) = sum((b-a)*g((b-a)*U+a))/(10^k);
end

semilogx(10.^(1:n),II,'*--')
xlabel('n')
grid on
ylim([0 ceil(1.1*max(II))])

%% Check answer using symbolic math
syms x
I = int(g(x),x,a,b)
double(I)
```

```
>> MonteCarloInt_Exer_a
I =
  40.0020
I =
40
ans =
   40
```




```
%% In-class Exercise (b)

close all; clear all;

g = @(x) exp(-x.^2); % Define the function to be integrated
a = 1; b = 3; % Limits for the integration

I = int(sym('exp(-x^2)'),a,b)
double(I)

n = 7;
UU = rand(1,10^n);

I = sum((b-a)*g((b-a)*UU+a))/(10^n)


%% Visualizing the convergence
II = zeros(1,n);
for k = 1:n
   U = UU(1:10^k);
   II(k) = sum((b-a)*g((b-a)*U+a))/(10^k);
end

semilogx(10.^(1:n),II,'*--')
xlabel('n')
grid on
ylim([0 (1.1*max(II))])

%% Check answer using symbolic math
syms x
I = int(g(x),x,a,b)
double(I)
```

```
>> MonteCarloInt_Exer_b
I =
   0.1395
I =
-(pi^(1/2)*(erf(1) - erf(3)))/2
ans =
   0.1394
```




```
close all; clear all;

g = @(x) exp(-(x.^2)); % Define the function to be integrated
a = 0; b = inf; % Limits for the integration

n = 7;
U = rand(1,10^n);

%b = 1/(a+1); a = 0;
I = mean((g(1./U - 1))./(U.^2))

%% Visualizing the convergence
UU = U;
II = zeros(1,n);
for k = 1:n
   U = UU(1:10^k);
   II(k) = sum((g(1./U - 1))./(U.^2))/(10^k);
end

semilogx(10.^(1:n),II,'*--')
xlabel('n')
grid on
ylim([0 ceil(1.1*max(II))])

%% Check answer using symbolic math
syms x
I = int(g(x),x,a,b)
double(I)
```

```
>> MonteCarloInt_Exer_c
I =
   0.8862
I =
pi^(1/2)/2
ans =
   0.8862
```